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Proof Theory for Minimal Quantum Logic I

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In this paper we give a sequential system of minimal quantum logic which enjoys cut-freeness naturally. The duality theorem, the cut-elimination theorem, and the completeness theorem with respect to the relational semantics of R. I. Goldblatt are presented. Due to severe limitations of space, technically heavy proofs of the first two theorems are relegated to a subsequent paper.

1. INTRODUCTION

More than a decade ago we introduced Gentzen's sequential methods into the realm of quantum logic (Nishimura, 1980). The method was elaborated further by Cutland and Gibbins (1982) to render a sequential system of minimal quantum logic which enjoys regularity and duality. Finally Tamura (1988) has succeeded in giving a cut-free sequential system of minimal quantum logic. However, his system enjoys cut-freeness in such an unreasonably distorted manner that his proof for the cut-elimination theorem appears more esoteric than it really is. The main purpose of the present paper is to present a more lucid sequential makeup of minimal quantum logic, and then to establish its fundamental properties, including the so-called cut-elimination theorem.

While in distributive logics the relationship of cut-free sequential systems such as LK and LJ and their semantics are so direct as to make the completeness proof an easy exercise, in nondistributive logics somewhat daunting proof-theoretic preliminary considerations seem inevitable to arrive finally at the completeness theorems of cut-free sequential systems. Although we deal exclusively with minimal quantum logic in this paper, our present study will hopefully shed light, as a prototype, on future study of nondistributive logics with an involutive negation in which de Morgan's

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laws hold. We expect in particular that this will be the case for so-called quantum logic in which the orthomodular law obtains. While the study of intermediate logics between classical and intuitionistic logics is flourishing, the study of intermediate logics between classical and minimal quantum logics remains relatively untouched. We hope that the present study will change the situation radically.

The sequential system GMQL that we now enunciate for minimal quantum logic consists of the following inference rules:

$$\frac{\Gamma \to \Delta}{\pi, \Gamma \to \Delta, \Sigma} \quad \text{(extension)}$$

$$\frac{\alpha, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}, \quad \frac{\beta, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta} \quad (\wedge \to)$$

$$\frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha \vee \beta}, \quad \frac{\Gamma \to \Delta, \beta}{\Gamma \to \Delta, \alpha \vee \beta} \quad (\to \vee)$$

$$\frac{\alpha \to \Delta \quad \beta \to \Delta}{\alpha \vee \beta \to \Delta} \quad (\vee \to), \quad \frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta} \quad (\to \wedge)$$

$$\frac{\Gamma \to \Delta}{\Delta', \Gamma \to} \quad ({}' \to), \quad \frac{\Gamma \to \Delta}{\to \Delta, \Gamma'} \quad (\to {}')$$

$$\frac{\alpha, \Gamma \to \Delta}{\alpha'', \Gamma \to \Delta} \quad ({}'' \to), \quad \frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha''} \quad (\to {}'')$$

$$\frac{\Gamma \to \Delta}{\Delta' \to \Gamma'} \quad ({}' \to {}')$$

$$\frac{\alpha', \Gamma \to \Delta}{(\alpha \vee \beta)', \Gamma \to \Delta}, \quad \frac{\beta', \Gamma \to \Delta}{(\alpha \vee \beta)', \Gamma \to \Delta} \quad (\vee' \to)$$

$$\frac{\Gamma \to \Delta, \alpha'}{\Gamma \to \Delta, (\alpha \wedge \beta)'}, \quad \frac{\Gamma \to \Delta, \beta'}{\Gamma \to \Delta, (\alpha \wedge \beta)'} \quad (\to \wedge')$$

$$\frac{\alpha' \to \Delta \quad \beta' \to \Delta}{(\alpha \wedge \beta)' \to \Delta} \quad (\vee \to'), \quad \frac{\Gamma \to \alpha' \quad \Gamma \to \alpha'}{\Gamma \to (\alpha \vee \beta)'} \quad (\to \vee')$$

$$\frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\alpha \vee \beta, \Gamma \to} \quad (\vee \to'), \quad \frac{\alpha' \to \Delta \quad \beta' \to \Delta}{\to \Delta, \alpha \wedge \beta} \quad ({}' \to \wedge)$$

Now some notational and terminological comments are in order. In this paper we adopt $'$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction) as primitive logical symbols. Propositional variables are denoted by $p, q, \ldots$, while wffs (well-formed formulas), also called formulas, are denoted by

$\alpha, \beta, \ldots$. The *grade* of a wff α, denoted by $g(\alpha)$, is defined inductively as follows:

(1) $g(p) = 0$ for any propositional variable p.
(2) $g(\alpha') = g(\alpha) + 1$.
(3) $g(\alpha \wedge \beta) = g(\alpha \vee \beta) = g(\alpha) + g(\beta) + 2$.

Finite (possibly empty) sets of wffs are denoted by $\Gamma, \Delta, \Pi, \ldots$. Given a finite set Γ of wffs, Γ' denotes the set $\{\alpha' | \alpha \in \Gamma\}$. A *sequent* $\Gamma \to \Delta$ means the ordered pair (Γ, Δ) of finite sets Γ and Δ, while the sets Γ and Δ are called the *antecedent* and the *succedent* of the sequent $\Gamma \to \Delta$, respectively. Such self-explanatory notations as $\Pi, \Gamma \to \Delta, \Sigma$ for $\Pi \cup \Gamma \to \Delta \cup \Sigma$ are used freely. A sequent of the form $\alpha \to \alpha$ is called an *axiom sequent*. The notion of a *proof* P of a sequent $\Gamma \to \Delta$ with *length* n is defined inductively as follows:

(1) Any axiom sequent $\alpha \to \alpha$ is a proof of itself with length 0.
(2) If P is a proof of a sequent $\Gamma \to \Delta$ with length n and

$$\frac{\Gamma \to \Delta}{\Pi \to \Sigma}$$

is an instance of an inference rule of GMQL, then

$$\frac{P}{\Pi \to \Sigma}$$

is a proof of the sequent $\Pi \to \Sigma$ with length $n + 1$.

(3) If P_i is a proof of a sequent $\Gamma_i \to \Delta_i$ with length n_i $(i = 1, 2)$ and

$$\frac{\Gamma_1 \to \Delta_1 \quad \Gamma_2 \to \Delta_2}{\Pi \to \Sigma}$$

is an instance of an inference rule of GMQL, then

$$\frac{P_1 \quad P_2}{\Pi \to \Sigma}$$

is a proof of the sequent $\Pi \to \Sigma$ with length $\max\{n_1, n_2\} + 1$.

The length of a proof P is denoted by $l(P)$. A sequent $\Gamma \to \Delta$ is said to be *provable* if it has a proof. Otherwise it is called *consistent*.

Although our cut-free sequential system GMQL does not satisfy the so-called subformula property in its strict sense, it gives a decision procedure for the word problem of free ortholattices once the completeness theorem is established, for which it suffices to note that $g(\alpha') < g((\alpha \wedge \beta)')$

and $\mathscr{g}(\beta') < \mathscr{g}((\alpha \wedge \beta)')$ for rule $(\wedge' \to)$ by way of example. For algebraic and semantical decision procedures, the reader is referred to Bruns (1976) and Goldblatt (1974, 1975). Fortunately, minimal quantum logic enjoys these three kinds of decision procedures; algebraic and semantical approaches to the decision problem of quantum logic have not so far succeeded. This is why we should try the third one, and this paper hopes to be the starting point of its proof-theoretic study.

Roughly speaking, if we deprive our system GMQL of the inference rules $(\vee' \to)$, $(\to \wedge')$, $(\wedge' \to)$, $(\to \vee')$, $(\vee \to')$, and $(' \to \wedge)$ and we agree to admit somewhat restricted (cut) as an inference rule, we obtain the system of Cutland and Gibbins (1982). The rule (cut) consists of the following two forms:

$$\frac{\Gamma \to \Delta_1, \alpha \quad \alpha \to \Delta_2}{\Gamma \to \Delta_1, \Delta_2} \qquad \text{(cut-1)}$$

$$\frac{\Gamma_1 \to \alpha \quad \alpha, \Gamma_2 \to \Delta}{\Gamma_1, \Gamma_2 \to \Delta} \qquad \text{(cut-2)}$$

The wff α in (cut-1) and (cut-2) is called the *cut formula*. In passing we note that to admit unrestricted (cut) for such logical systems as ours would make them degenerate into classical logic, as was remarked by Cutland and Gibbins (1982).

Tamura (1988) gave his cut-free system by exploiting the legacy of Cutland and Gibbins (1982) but incorporating their inference rules surely except (cut) into his system in unnecessarily restricted forms. This unreasonable restriction forced him in the proof of the cut-elimination theorem to combine wffs in the antecedent of a sequent by conjunction and wffs in its succedent by disjunction, and then to dissolve such unnatural combinations. Such a proof is not compatible with Gentzen's (1935) original philosophy and aesthetics, and is to be avoided if possible.

Furthermore, the conceptual significance of Lemma 4 in Tamura's (1988) paper remained vague at best there. This is distilled into the duality theorem in Section 2, which is followed by the so-called cut-elimination theorem in Section 3. We believe that the duality theorem is no less important than the cut-elimination theorem itself, and we would like to propose that these two theorems should be called the first and the second fundamental theorems of proof theory of GMQL. The final section is devoted to the completeness theorem. A proof for the completeness theorem without recourse to the cut-elimination theorem, which would give a semantical proof for the cut-elimination, seems an intriguing topic for future study.

2. THE DUALITY THEOREM

Two wffs α and β are said to be *provably equivalent*, in notation $\alpha \simeq \beta$, if for any finite sets Γ and Δ of wffs we have that

(a) the sequent $\alpha, \Gamma \to \Delta$ is provable iff the sequent $\beta, \Gamma \to \delta$ is provable; and
(b) the sequent $\Gamma \to \Delta, \alpha$ is provable iff the sequent $\Gamma \to \Delta, \beta$ is provable.

It is easy to see that this is indeed an equivalence relation among wffs.

Theorem 2.1 The fundamental theorem of provability equivalence). If $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$, *then* $\alpha'_1 \sim \beta'_1$, $\alpha_1 \wedge \alpha_2 \simeq \beta_1 \wedge \beta_2$, and $\alpha_1 \vee \alpha_2 \simeq \beta_1 \vee \beta_2$.

Proof. If $\gamma, \delta_1, \ldots, \delta_n$ are wffs and $p_1, \ldots, p_n$ are distinct propositional variables, we write $\gamma[\delta_1/p_1, \ldots, \delta_n/p_n]$ for the wff obtained from γ by replacing every occurrence of p_i by δ_i $(1 < i < n)$. Whenever we use this notation, it will always be assumed that the propositional variables at issue are distinct. The theorem follows readily from the following two statements

(I) If $\delta_1 \simeq \sigma_1, \ldots, \delta_n \simeq \sigma_n$ and a sequent $\gamma[\delta_1/p_1, \ldots, \delta_n/p_n]$, $\Gamma \to \Delta$ has a proof P with $l(P) \leq m$, then the sequent $\gamma[\sigma_1/p_1, \ldots, \sigma_n/p_n]$, $\Gamma \to \Delta$ is also provable.

(II) If $\delta_1 \simeq \sigma_1, \ldots, \delta_n \simeq \sigma_n$, and a sequent $\Gamma \to \Delta, \gamma[\delta_1/p_1, \ldots, \delta_n/p_n]$ has a proof P with $l(P) \leq m$, then the sequent $\Gamma \to \Delta, \gamma[\sigma_1/p_1, \ldots, \sigma_n/p_n]$ is also provable.

These two statements are proved simultaneously by double induction principally on $g(\gamma)$ and secondly on m. The proof is divided into cases according to which inference rule is used as the last inference in P. The details are safely left to the reader.

The lengthy proof of the following first main theorem of this section is relegated to a subsequent paper.

Theorem 2.2 (The first duality theorem). If $\alpha \simeq \beta$, then $\alpha \simeq \beta''$.

Theorem 2.2 implies a version of Lemma 4 of Tamura (1988) at once.

Corollary 2.3. (a) If a sequent $\Gamma, \Pi' \to \Delta, \Sigma'$ is provable, then the sequent $\Delta', \Sigma \to \Gamma', \Pi$ is also provable.

(b) If a sequent $\Gamma \to \Delta'$ is provable, then the sequent $\Gamma, \Delta \to$ is provable.

(c) If a sequent $\Gamma' \to \Delta$ is provable, then the sequent $\to \Gamma, \Delta$ is also provable.

Proof. By Theorem 2.2 it suffices only to take into account the rules $(' \to ')$, $(' \to)$, and $(\to ')$.

The proof of the following main theorem of this section is also deferred to a subsequent paper.

Theorem 2.4 (The second duality theorem). If $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$, then $\alpha_1 \wedge \alpha_2 \simeq (\beta_1' \vee \beta_2')'$ and $\alpha_1 \vee \alpha_2 \simeq (\beta_1' \wedge \beta_2')'$.

Corollary 2.5. If $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$, then $\alpha_1' \wedge \alpha_2' \simeq (\beta_1 \vee \beta_2)'$ and $\alpha_1' \vee \alpha_2' \simeq (\beta_1 \wedge \beta_2)'$.

Proof. By Theorem 2.1, 2.2, and 2.4, we have that $\alpha_1' \wedge \alpha_2' \simeq (\alpha_1'' \vee \alpha_2'')' \simeq (\beta_1 \vee \beta_2)'$ and $\alpha_1' \vee \alpha_2' \simeq (\alpha_1'' \wedge \alpha_2'')' \simeq (\beta_1 \wedge \beta_2)'$.

3. THE CUT-ELIMINATION THEOREM

Theorem 3.1. A sequent $\alpha, \beta, \Gamma \rightarrow \Delta$ is provable iff the sequent $\alpha \wedge \beta, \Gamma \rightarrow \Delta$ is provable. Similarly, a sequent $\Pi \rightarrow \Sigma, \gamma, \delta$ is provable iff the sequent $\Pi \rightarrow \Sigma, \gamma \vee \delta$ is provable.

Proof. For both statements, the only-if part follows readily from $(\wedge \rightarrow)$ or $(\rightarrow \vee)$. The if part can be established by induction on the construction of a proof of $\alpha \wedge \beta, \Gamma \rightarrow \Delta$ or $\Gamma \rightarrow \Delta, \alpha \vee \beta$.

Corollary 3.2. A sequent $\alpha', \beta', \Gamma \rightarrow \Delta$ is provable iff the sequent $(\alpha \vee \beta)', \Gamma \rightarrow \Delta$ is provable. Similarly, a sequent $\Pi \rightarrow \Sigma, \gamma', \delta'$ is provable iff the sequent $\Pi \rightarrow \Sigma, (\gamma \wedge \delta)'$ is provable.

Proof. Follows from Corollary 2.5 and Theorem 3.1.

Theorem 3.3. If a sequent $\alpha \vee \beta, \Gamma \rightarrow \Delta$ is provable, then the sequents $\alpha, \Gamma \rightarrow \Delta$ and $\beta, \Gamma \rightarrow \Delta$ are provable. Similarly, if a sequent $\Pi \rightarrow \Sigma, \gamma \wedge \delta$ is provable, then the sequents $\Pi \rightarrow \Sigma, \gamma$ and $\Pi \rightarrow \Sigma, \delta$ are provable.

Proof. By induction on the construction of a proof of $\alpha \vee \beta, \Gamma \rightarrow \Delta$ or $\Pi \rightarrow \Sigma, \gamma \wedge \delta$. Here we deal only with the case that the last step of a proof of a sequent $\alpha \vee \beta, \Gamma \rightarrow \Delta$ is $(\vee \rightarrow')$. So it must be one of the following two forms.

$$\frac{\Gamma \rightarrow \alpha' \quad \Gamma \rightarrow \beta'}{\alpha \vee \beta, \Gamma \rightarrow} \quad (\vee \rightarrow')$$

$$\frac{\alpha \vee \beta, \Gamma_1 \rightarrow \sigma' \quad \alpha \vee \beta, \Gamma_1 \rightarrow \rho'}{\alpha \vee \beta, \sigma \vee \rho, \Gamma_1 \rightarrow} \quad (\vee \rightarrow')$$

In the former case the sequents $\alpha'', \Gamma \rightarrow$ and $\beta'', \Gamma \rightarrow$ are provable by $(' \rightarrow)$. So the desired sequents $\alpha, \Gamma \rightarrow$ and $\beta, \Gamma \rightarrow$ are provable by Theorem 2.2. In the latter case the sequents $\alpha, \Gamma_1 \rightarrow \sigma'$, $\beta, \Gamma_1 \rightarrow \sigma'$, $\alpha, \Gamma_1 \rightarrow \rho'$, and $\beta, \Gamma_1 \rightarrow \rho'$, are provable by the induction hypothesis. So the desired sequents $\alpha, \sigma \vee \rho, \Gamma_1 \rightarrow$ and $\beta, \sigma \wedge \rho, \Gamma_1 \rightarrow$ are provable as follows:

$$\frac{\alpha, \Gamma_1 \rightarrow \sigma' \quad \alpha, \Gamma_1 \rightarrow \rho'}{\alpha, \sigma \vee \rho, \Gamma_1 \rightarrow} \quad (\vee \rightarrow')$$

$$\frac{\beta, \Gamma_1 \rightarrow \sigma' \quad \beta, \Gamma_1 \rightarrow \rho'}{\beta, \sigma \vee \rho, \Gamma_1 \rightarrow} \quad (\vee \rightarrow')$$

Corollary 3.4. If a sequent $(\alpha \wedge \beta)', \Gamma \rightarrow \Delta$ is provable, then the sequents $\alpha', \Gamma \rightarrow \Delta$ and $\beta', \Gamma \rightarrow \Delta$ are provable. Similarly, if a sequent $\Pi \rightarrow \Sigma, (\gamma \vee \delta)'$ is provable, then the sequents $\Pi \rightarrow \Sigma, \gamma'$ and $\Pi \rightarrow \Sigma, \delta'$ are provable.

Proof. Follows from Theorem 3.3 and Corollary 2.5.

The lengthy proof of the following main theorem of this section is postponed to a subsequent paper.

Theorem 3.5 (The cut-elimination theorem). If sequents $\Gamma_1 \rightarrow \Delta_1, \alpha$ and $\alpha, \Gamma_2 \rightarrow \Delta_2$ are provable with $\Delta_1 = \emptyset$ or $\Gamma_2 = \emptyset$, then the sequent $\Gamma_1, \Gamma_2 \rightarrow \Sigma_1, \Delta_2$ is also provable. In other words, (cut) is permissible in GMQL.

4. THE COMPLETENESS THEOREM

An *O-frame* is a pair $(X, {}^\perp)$ of a nonempty set X and an orthogonality relation (i.e., an irreflexive and symmetric binary relation) on X. Given $Y \subseteq X$, we write $Y^\perp$ for the set $\{x \in X | x^\perp y \text{ for any } y \in Y\}$. A subset Y of X is said to be ${}^\perp$-closed if $Y = Y^{\perp\perp}$.

An *O-model* is a triple $(X, {}^\perp, D)$, where $(X, {}^\perp)$ is an *O-frame* and D assigns to each propositional variable p a ${}^\perp$-closed subset $D(p)$ of X. The notation $\|\alpha\|$ for a wff α is defined inductively as follows:

(1) $\|p\| = D(p)$ for any propositional variable p.
(2) $\|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\|$.
(3) $\|\alpha'\| = \|\alpha\|^\perp$.
(4) $\|\alpha \vee \beta\| = \|\alpha' \wedge \beta'\| = (\|\alpha\|^\perp \cap \|\beta\|^\perp)^\perp$.

Given $x \in X$ and a wff α, we write $V(\alpha; x) = 1$ if $x \in \|\alpha\|$ and $V(\alpha; x) = 0$ if $x \notin \|\alpha\|$. Given $x \in X$ and a sequent $\Gamma \rightarrow \Delta$, we write $V(\Gamma \rightarrow \Delta; x) = 1$ if $x \in \bigcap \{\|\alpha\| \mid \alpha \in \Gamma\}$ and $x \notin (\cup\{\|\beta\|^\perp | \beta \in \Delta\})^\perp$, and $V(\Gamma \rightarrow \Delta; x) = 0$ otherwise.

A sequent $\Gamma \rightarrow \Delta$ is said to be *realizable* if there exists an *O-model* $(X, {}^\perp, D)$ and some $x \in X$ such that $V(\Gamma \rightarrow \Delta; x) = 1$. The sequent $\Gamma \rightarrow \Delta$ is called *valid* otherwise.

Theorem 4.1 (The soundness theorem). If a sequent $\Gamma \rightarrow \Delta$ is provable, then it is valid.

Proof. By induction on the construction of a proof of the sequent $\Gamma \rightarrow \Delta$.

A set Ω of wffs is said to be *admissible* if it satisfies the following conditions:

(1) If p is a propositional variable and $p \in \Omega$, then $p' \in \Omega$.
(2) If $\alpha \in \Omega$ and β is a subformula of α, then $\beta \in \Omega$.
(3) If $(\alpha \vee \beta) \in \Omega$, then $(\alpha' \wedge \beta')' \in \Omega$.

A finite set Γ of wffs is said to be *inconsistent* if for some wff α, both of the sequents $\Gamma \rightarrow \alpha$ and $\Gamma \rightarrow \alpha'$ are provable. Otherwise the set Γ is said to be *consistent.*

Lemma 4.2. A finite set Γ of wffs is inconsistent iff the sequent $\Gamma \rightarrow$ is provable.

Proof. The if part is obvious. The only-if part can be shown easily as follows:

$$\dfrac{\dfrac{\Gamma \rightarrow \alpha \quad \Gamma \rightarrow \alpha'}{\Gamma \rightarrow \alpha \wedge \alpha'}(\rightarrow \wedge) \qquad \dfrac{\dfrac{\alpha \rightarrow \alpha}{\alpha, \alpha' \rightarrow}(' \rightarrow)}{\alpha \wedge \alpha' \rightarrow}(\wedge \rightarrow)}{\Gamma \rightarrow}(\text{cut})$$

Given an admissible set Ω of wffs, the Ω-*canonical* O-model $\mathcal{M}(\Omega) = (X_\Omega, \perp_\Omega, D_\Omega)$ is defined as follows:

(1) X_Ω is the set of all the consistent subsets of Ω.
(2) For any $\Gamma_1, \Gamma_2 \in X_\Omega$, $\Gamma_1 \perp_\Omega \Gamma_2$ iff for some $\alpha' \in \Omega$, either: (a) both of the sequents $\Gamma_1 \rightarrow \alpha$ and $\Gamma_1 \rightarrow \alpha'$ are provable, or (b) both of the sequents $\Gamma_1 \rightarrow \alpha'$ and $\Gamma_2 \rightarrow \alpha$ are provable.
(3) If $p \notin \Omega$, then $D_\Omega(p) = \varnothing$, while if $p \in \Omega$, then $D_\Omega(p)$ consists of all the consistent subsets Γ of Ω such that the sequent $\Gamma \rightarrow p$ is provable.

Theorem 4.3. $\mathcal{M}(\Omega)$ is an O-model.

Proof. Obviously the relation $\perp_\Omega$ is symmetric. That the relation $\perp_\Omega$ is irreflexive follows from our assumption that every element of X_Ω is a consistent set of wffs. Now it remains to show that $D_\Omega(p)$ is $\perp_\Omega$-closed for any propositional variable p. Unless $p \in \Omega$, there is nothing to prove. So let $p \in \Omega$. Let Γ be an element of X_Ω such that the sequent $\Gamma \rightarrow p$ is not provable. Suppose for the sake of contradiction that the set $\{p'\}$ is inconsistent, which implies by Lemma 4.2 that the sequent $p' \to$ is provable. By Corollary 2.3 the sequent $\to p$ is provable, which implies by (extension) that the sequent $\Gamma \to p$ is provable. This is a contradiction. So $\{p'\} \in X_\Omega$. Suppose, for the sake of contradiction, that for some $\alpha' \in \Omega$, either both of the sequents $\Gamma \to \alpha$ and $p' \to \alpha$ are provable or both of the sequents $\Gamma \to \alpha'$ and $p' \to \alpha$ are provable. Here we deal only with the former case, leaving a similar treatment of the latter to the reader. By Corollary 2.3 the sequent $\alpha \to p$ is provable, which implies by (cut) that the sequent $\Gamma \to p$ is provable. This is a contradiction. Thus it cannot be the case that $\Gamma \perp_\Omega \{p'\}$, while for any $\Delta \in X_\Omega$ such that the sequent $\Delta \to p$ is provable, $\Delta \perp_\Omega \{p\}$. This implies that the set of all $\Delta \in X_\Omega$ such that the sequent $\Delta \to p$ is provable is $\perp_\Omega$-closed.

The *disjunction grade* of a wff α, denoted by $g_\vee(\alpha)$, is defined inductively as follows:

(1) $g_\vee(p) = 0$ for any propositional variable p.
(2) $g_\vee(\alpha') = g_\vee(\alpha)$.
(3) $g_\vee(\alpha \wedge \beta) = g_\vee(\alpha) + g_\vee(\beta)$.
(4) $g_\vee(\alpha \vee \beta) = g_\vee(\alpha) + g_\vee(\beta) + 1$.

Theorem 4.4 (The fundamental theorem for $\mathcal{M}(\Omega)$). For any $\alpha \in \Omega$ and any $\Gamma \in X_\Omega$, the sequent $\Gamma \to \alpha$ is provable iff $\Gamma \in \|\alpha\|$ in $\mathcal{M}(\Omega)$.

Proof. The proof is carried out by double induction principally on $g_\vee(\alpha)$ and secondarily on $g(\alpha)$. The proof is divided into several cases.

(a) In the case that α is a propositional variable: It follows from the definition of D_Ω.

(b) In the case that $\alpha = \beta'$ for some wff β: If $\Gamma \to \beta'$ is provable, then $\Gamma \perp_\Omega \|\beta\|$ by induction hypothesis, which implies that $\Gamma \in \|\beta'\|$. Suppose, for the sake of contradiction, that the set$\{\beta\}$ is inconsistent, which implies by Lemma 4.2 that the sequent $\beta \to$ is provable. Thus the sequent $\Gamma \to \beta'$ is provable as follows:

$$\cfrac{\cfrac{\beta \to}{\to \beta'}\ (\to')}{\Gamma \to \beta'}\ (\text{extension})$$

This is a contradiction. So it must be the case that $\{\beta\} \in X_\Omega$. Suppose, for the sake of contradiction, that for some $\gamma' \in \Omega$, either both of the sequents $\Gamma \to \gamma'$ and $\beta \to \gamma$ are provable or both of the sequents $\Gamma \to \gamma$ and $\beta \to \gamma'$ are provable. Here we deal only with the former case, leaving safely a similar treatment of the latter to the reader. The desired contradiction is obtained as follows:

$$\cfrac{\Gamma \to \gamma' \qquad \cfrac{\beta \to \gamma}{\gamma' \to \beta'}\ ('\to')}{\Gamma \to \beta'}\ (\text{cut})$$

Thus it cannot be the case that $\Gamma \perp_\Omega \{\beta\}$. Since $\{\beta\} \in \|\beta\|$ by induction hypothesis, this means that $\Gamma \notin \|\beta\|^{\perp}\Omega = \|\beta'\|$.

(c) In the case that α is of the form $\beta \wedge \gamma$ for some wffs β and γ: If the sequent $\Gamma \to \alpha$ is provable, then both of the sequents $\Gamma \to \beta$ and $\Gamma \to \gamma$ are provable by Theorem 3.3, which implies by induction hypothesis that $\Gamma \in \|\beta\|$ and $\Gamma \in \|\gamma\|$. So $\Gamma \in \|\beta\| \cap \|\gamma\| = \|\beta \wedge \gamma\|$. Unless the sequent $\Gamma \to \alpha$ is provable, suppose, for the sake of contradiction, that both of the sequents $\Gamma \to \beta$ and $\Gamma \to \gamma$ are provable. The desired conclusion is obtained as follows:

$$\frac{\Gamma \to \beta \qquad \Gamma \to \gamma}{\Gamma \to \beta \wedge \gamma} \quad (\to \wedge)$$

Thus one of the sequents $\Gamma \to \beta$ and $\Gamma \to \gamma$ is consistent, which implies by induction hypothesis that $\Gamma \notin \|\beta\|$ or $\Gamma \notin \|\gamma\|$. So $\Gamma \notin \|\beta \wedge \gamma\| = \|\beta\| \cap \|\gamma\|$.

(d) In the case that α is of the form $\beta \vee \gamma$ for some wffs β and γ: Use Theorem 2.4.

Theorem 4.5 (The completeness theorem). A sequent $\Gamma \to \Delta$ is realizable iff it is consistent.

Proof. The only-if part is the soundness theorem already established. To see the if part, take an admissible set Ω such that $\Gamma \cup \{\beta_1 \vee \cdots \vee \beta_n\} \subseteq \Omega$, where $\Delta = \{\beta_1, \ldots, \beta_n\}$. By Theorem 3.1 the sequent $\Gamma \to \Delta$ is consistent iff the sequent $\Gamma \to \beta_1 \vee \cdots \vee \beta_n$ is consistent. The desired conclusion follows readily from Theorem 4.4.

We remark in passing that in the proof of Theorem 4.5 it does not matter how to insert parentheses in $\beta_1 \vee \cdots \vee \beta_n$.

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